Energy level statistics of Andreev quasiparticles

1995 J. Phys.: Condens. Matter 7 4105

(http://iopscience.iop.org/0953-8984/7/21/010)

J. Phys.: Condens. Matter 7 (1995) 4105–4113. Printed in the UK

Energy level statistics of Andreev quasiparticles

Richard Berkovits

The Jack and Pearl Resnick Institute of Advanced Technology, Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

Received 3 January 1995

Abstract. The behaviour of the low-lying quasiparticle energy levels in a superconductor–normal metal–superconductor structure is investigated analytically and numerically. The energy levels are periodic in the order parameter phase difference between the superconducting regions with a period of 2π. The statistical properties of the spectra exhibit a transition from GOE to GUE statistics as a function of the phase difference with a period of π.

Understanding the energy level statistics of normal mesoscopic systems has been in the centre of an ongoing research effort [1–7] Most of the statistical properties can be explained in the framework of the random matrix theory (RMT) originally proposed by Wigner and Dyson to describe the spectrum properties of complex nuclei [8–10] Metallic systems which have time reversal symmetry obey the Gaussian orthogonal matrix ensemble (GOE) eigenvalue statistics, while for systems where time reversal symmetry is broken (for example by a magnetic field) unitary matrix ensemble (GUE) eigenvalue statistics emerges. Symplectic ensemble (GSE) statistics is observed for cases in which spin–orbit scattering is present. On the other hand, once the system is localized the spectrum follows the Poisson statistics.

Crossover between the different statistical ensembles is the subject of many recent studies [11–20] In the disorder-induced metal–insulator transition a crossover between GOE and Poisson statistics occurs [11–13] and a new dimensional dependent statistical behaviour of the spectrum at the mobility edge is predicted [14, 15]. The study of the transition from the GOE to GUE statistics has concentrated on the effect of a magnetic field [16–20] has been shown that even for strong magnetic fields it is hard to obtain clean GUE characteristics of the spectrum [18–20] due to short paths for which the time reversal symmetry is not completely broken.

Another set of systems for which the time reversal symmetry is broken are the hybrid normal metal–superconductor structures. These structures have recently received much attention [21–27] For the most part, experiment and theory have concentrated on their transport properties. In this paper we shall consider the statistical properties of the spectrum of these systems.

As a particular example the superconductor–normal metal–superconductor (SNS) junction shown in figure 1 will be considered. The system is sensitive to the order parameter phases χ_1 and χ_2 in the superconducting regions due to the Andreev reflections on the normal metal–superconducting (NS) boundary [28]. An electron above the Fermi energy (or a hole below the Fermi energy) reflected from the NS boundary is transformed into a hole (electron) and acquires an additional phase of $-\chi_{1,2}$. Once the phases the two superconducting regions are

0953-8984/95/214105+09$19.50

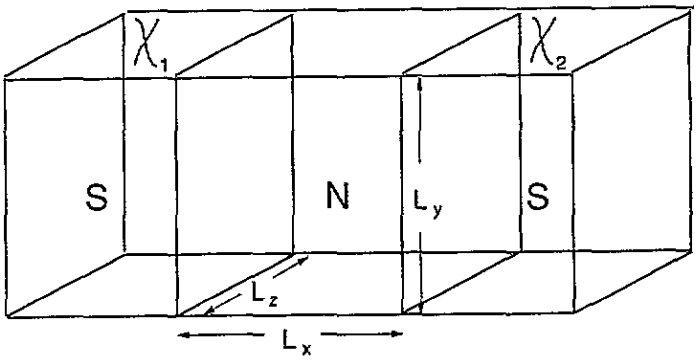

Figure 1. *A SNS junction with an order parameter $\Delta_0 \exp(i\chi_1)$ on the left superconducting* region and $\Delta_0 \exp(i\chi_2)$ on the right superconducting region. The normal region has dimensions of $L_x \times L_y \times L_z$.

not equal this will lead to a broken time reversal symmetry. These quasiparticles excitations are described by positive eigenvalues of the Bogoliubov–de Gennes (BdG) equation [29]

$$\begin{pmatrix} H & \Delta \\ \Delta^* & -H \end{pmatrix} \Psi = \varepsilon \Psi \tag{1}$$

where $\Psi(r)$ is a two-component wave function and $H = p^2/2m + V(r) - E_F$ is the single-electron Hamiltonian for a given potential V. In the superconducting region $V(|x| > L_x/2) = 0$, while in the normal region $V(|x| < L_x/2)$ is assumed to be a Gaussian-distributed white noise, defined by

$$\begin{aligned} &\langle V(r) \rangle = 0 \\ &\langle V(r)V(r') \rangle = \gamma\, \delta(r - r') \end{aligned} \tag{2}$$

where $\langle \ldots \rangle$ denotes averaging over different realizations of the disorder potential, $\gamma = \hbar v_F/2\pi \ell N(E_F)$, $N(E_F)$ is the averaged density of states at the Fermi energy E_F, v_F is the Fermi velocity and ℓ is the elastic mean free path. The order parameter $\Delta(r)$ in the superconducting regions is approximated by $\Delta = \Delta_0 \exp(i\chi_1)$ in the first superconducting region ($x < -L_x/2$) and by $\Delta = \Delta_0 \exp(i\chi_2)$ in the second superconducting region ($x > L_x/2$), while in the normal part $\Delta = 0$. Thus, the reduction in the superconducting order parameter on approaching the SN interface is neglected. This is reasonable while considering low-lying excitation (i.e., $\varepsilon \ll \Delta_0$) confined to the normal region by the Andreev scattering. All the energies are measured relative to the Fermi energy.

In this paper the sensitivity of the variance in the number of low-lying quasiparticle states for a given energy window will be calculated using a diagrammatic perturbation method. The variance depends on the order parameter phase difference $\varphi = \chi_2 - \chi_1$, and oscillates with a period π, while the quasiparticle energy levels have a period of 2π. This is similar to the situation for the conductance of a diffusive SNS junction [21, 22]. For $\varphi = 0$ the variance corresponds to GOE statistics while for $\varphi = \pi/2$ it corresponds to GUE statistics. For values of φ in between those two values the variance follows a combination of both statistical ensembles. In order to verify those theoretical results we have performed exact diagonalization numerical calculations of a tight-binding BdG model. The results fit rather well to the theory. We have also numerically calculated the level spacing distribution and found that it is less sensitive to φ than the variance of the level number. These properties can be investigated experimentally by tunnelling experiments through the SNS junction or by far-infrared spectroscopy experiments of an array on such junctions.

The number of states for a particular realization of disorder in a given energy range is

$$N_\varepsilon(E) = -\frac{1}{\pi}\int_{\varepsilon-E/2}^{\varepsilon+E/2} d\varepsilon' \ \mathrm{Im} \int dr \ G^R(r, r, \varepsilon') \tag{3}$$

where the two-component Green function is given by

$$G^R(r, r', \varepsilon) = \sum_\mu \frac{\Psi_\mu^\dagger(r)\Psi_\mu(r')}{\varepsilon - \varepsilon_\mu + i\Gamma} \tag{4}$$

where Ψ_μ are the eigenvectors and ε_μ are the eigenvalues of the BdG equation in the presence of disorder. Γ corresponds to the inelastic level broadening.

Following Altshuler and Shklovskii [2] the variance in the number of levels for a given energy region may be written as

$$\langle \delta^2 N_\varepsilon(E)\rangle = \langle N_\varepsilon^2(E)\rangle - \langle N_\varepsilon(E)\rangle^2 = \int_{\varepsilon-E/2}^{\varepsilon+E/2} d\varepsilon_1 \ d\varepsilon_2 \ K(\varepsilon_1, \varepsilon_2) \tag{5}$$

where

$$K(\varepsilon_1, \varepsilon_2) = \frac{1}{\pi^2}\int dr_1 \ dr_2 \ [\langle \mathrm{Im}\, G^R(r_1, r_1, \varepsilon_1)\mathrm{Im}\, G^R(r_2, r_2, \varepsilon_2)\rangle - \langle \mathrm{Im}\, G^R(r_1, r_1, \varepsilon_1)\rangle\langle \mathrm{Im}\, G^R(r_2, r_2, \varepsilon_2)\rangle]. \tag{6}$$

This correlation function can be calculated using a diagrammatic impurity technique. The main contribution comes from the diagrams shown in figure 2 (see [2]), which correspond to

$$K(\varepsilon_1, \varepsilon_2) = \frac{1}{2\pi^2}\mathrm{Re}\int dr_1 \ dr_2 \ [(D(r_1, r_2, \varepsilon_1 - \varepsilon_2))^2 + (C(r_1, r_2, \varepsilon_1 - \varepsilon_2))^2] \tag{7}$$

where the diffusion propagator D and the cooperon C for the low-lying Andreev excitation have already been calculated in [21, 22] and have the following form in the momentum representation:

$$D(q, \varepsilon) = \left[D\left(\left(\frac{\pi}{L_x}n_x\right)^2 + \left(\frac{\pi}{L_y}n_y\right)^2 + \left(\frac{\pi}{L_z}n_z\right)^2\right) - i\varepsilon + \Gamma\right]^{-1}$$

$$C(q, \varepsilon) = \left[D\left(\left(\frac{\pi}{L_x}\left(n_x + \frac{\varphi}{\pi}\right)\right)^2 + \left(\frac{\pi}{L_y}n_y\right)^2 + \left(\frac{\pi}{L_z}n_z\right)^2\right) - i\varepsilon + \Gamma\right]^{-1} \tag{8}$$

where $n_x, n_y, n_z = 0, \pm1, \pm2, \ldots$. As can be seen the diffusion propagator is not affected by the order parameter phases while the cooperon is an oscillating function of φ with a period π. This period 'halving' is similar to the one seen for systems with an Aharonov–Bohm flux [16] and stems from the fact that the cooperon sums up the momenta of two particles. Inserting the diffusion and the cooperon into the correlation function (equation (7)) and then performing the integrations in equation (5) will lead to

$$\langle \delta^2 N_\varepsilon(E)\rangle = \frac{1}{2\pi^2}\sum_{n_x,n_y,n_z}\left[\ln\left(\left(\frac{E}{D\{(\pi n_x/L_x)^2(\pi n_y/L_y)^2 + (\pi n_z/L_z)^2\} + \Gamma}\right)^2 + 1\right) + \ln\left(\left(\frac{E}{D\{((\pi n_x + \varphi)/L_x)^2 + (\pi n_y/L_y)^2 + (\pi n_z/L_z)^2\} + \Gamma}\right)^2 + 1\right)\right]. \tag{9}$$

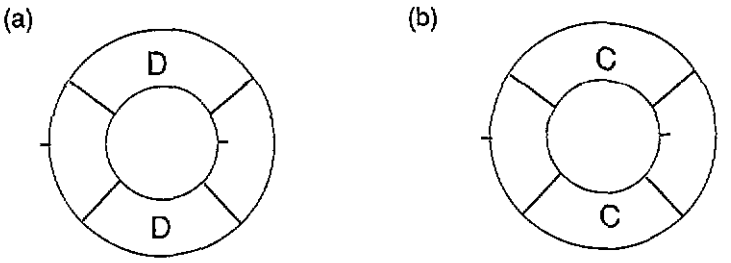

Figure 2. The main diagrams contributing to the correlation function $K(\varepsilon_1, \varepsilon_2)$. D represents a diffusion propagator and C a cooperon defined in equation (8).

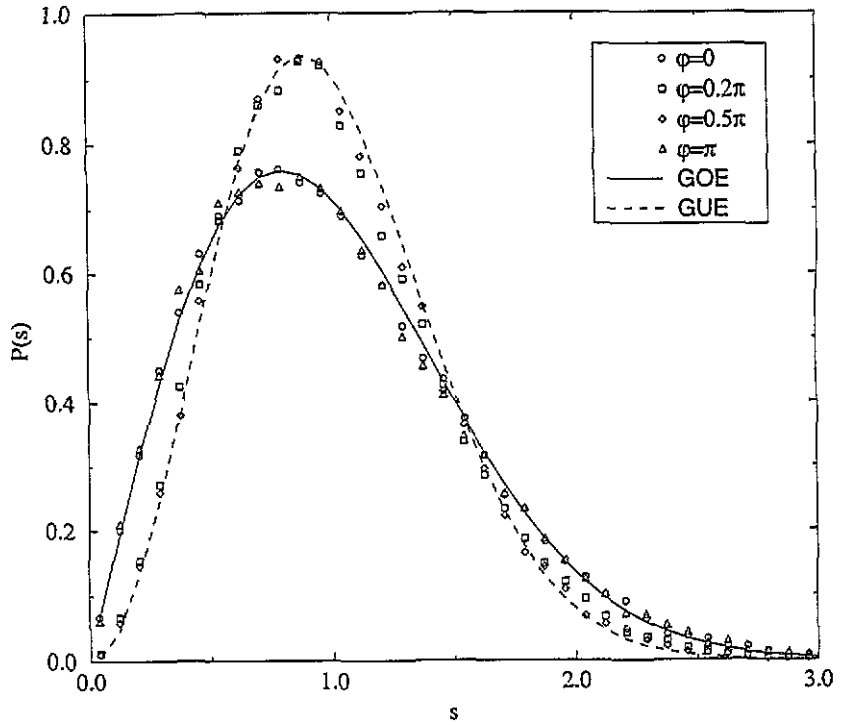

Figure 3. The level spacing distribution $P(s)$ averaged over the 30 lowest electronic levels for different values of the order parameter phase difference φ. The full line represents the GOE level spacing distribution, while the dashed line represents the GUE level spacing distribution.

The summation may be evaluated at different limits of E. For $E \ll D/L^2_{x,y,z}$ (i.e., an energy window smaller than the Thouless energy) the zero mode gives the main contribution

$$\langle \delta^2 N_\varepsilon(E) \rangle = \frac{1}{2\pi^2} \left[\ln\left(\frac{E^2}{\Gamma^2} + 1\right) + \ln\left(\frac{E^2}{(D(\tilde{\varphi}/L_x)^2 + \Gamma)^2} + 1\right)\right] \qquad (10)$$

where $\tilde{\varphi}^2 = \min\{(\pi n_x + \varphi)^2\}$. It is important to note that in the diagrammatic calculation a divergence occurs as the inelastic broadening $\Gamma \to 0$. This stems from the breakdown of the diagrammatic approximation for energy scales smaller than the averaged level separation [2] s. As long as one deals with $E \gg s$, one should set $\Gamma = s$ and the above results are

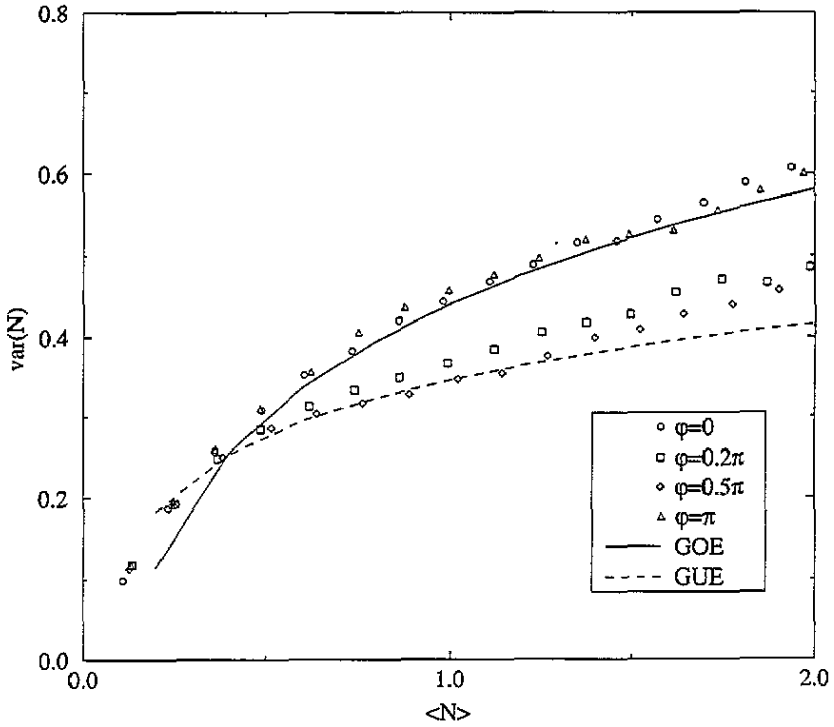

Figure 4. The variance of the number of levels in a given energy region as a function of the averaged number of levels for values of energy smaller than the Thouless energy. The full line represents the GOE variance while the dashed line represents the GUE variance given in equation (11).

similar to the results of the RMT theory [9]

$$\begin{aligned}\langle\delta^2 N_\varepsilon(E)\rangle_{\mathrm{GOE}} &= \frac{2}{\pi^2}\ln(\langle N_\varepsilon(E)\rangle) + \lambda_{\mathrm{GOE}} \\ \langle\delta^2 N_\varepsilon(E)\rangle_{\mathrm{GUE}} &= \frac{1}{\pi^2}\ln(\langle N_\varepsilon(E)\rangle) + \lambda_{\mathrm{GUE}}\end{aligned} \tag{11}$$

(where $\lambda_{\mathrm{GOE}} = 0.44$ and $\lambda_{\mathrm{GUE}} = 0.346$) up to the constants λ. Thus, for $\varphi = 0, \pi$ the variance of the number of energy levels for a given energy window E presented in equation (10) corresponds to the GOE results, while for $\varphi = \pi/2$ the variance is close to the GUE results. This is not surprising since for $\varphi = 0$ time reversal symmetry is not broken while for $\varphi = \pi$ the BdG equation exhibits only a false time reversal breaking [30, 16]. For other values of φ the time reversal symmetry is broken and a crossover towards GUE statistics is observed.

For $E \gg D/L^2_{x,y,z}$ one can exchange the summation in equation (9) with an integration [2] and thus obtain

$$\langle\delta^2 N_\varepsilon(E)\rangle = N_d V \left(\frac{E}{D}\right)^{d/2} \tag{12}$$

where d is the dimensionality and V is the volume of the normal part of the junction. N_d is a numerical factor: for the three-dimensional case $N_3 = \sqrt{2}/6\pi^3$ and for the two-dimensional case $N_2 = 1/4\pi^2$. It can be seen that for large values of E the variance is independent of the order parameter phase factor φ. Thus, the time reversal symmetry has no influence

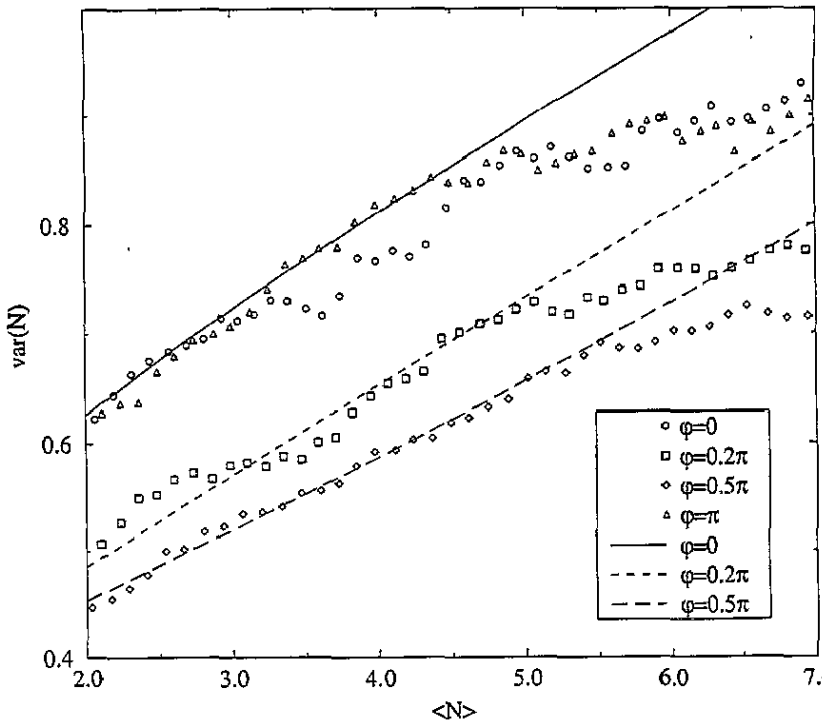

Figure 5. The variance of the number of levels for values of energy larger than the Thouless energy. The curves represent equation (9) with the corresponding values of φ.

on the variance at values of E larger than the Thouless energy. This is in contrast to the situation for a magnetic field in which there is a factor of two difference in the value of N_d between the case with a magnetic field and the case without one [2]. The difference stems from the fact that high values of the magnetic field nullify the cooperon while because of the oscillatory nature of the order parameter phase the cooperon remains always finite. This is similar to the situation discussed by Dupuis and Montambaux [16] for a ring threaded by an Aharonov–Bohm flux.

Until now we have discussed averaging over impurity realizations. Since when one prepares a new SNS junction realization the order parameter phases may also change, it makes sense to average also over the phases. Therefore, we define the phase-averaged variance as

$$\langle \delta^2 N_\varepsilon(E) \rangle_\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\varphi \, \langle \delta^2 N_\varepsilon(E) \rangle. \tag{13}$$

Inserting the value of $\langle \delta^2 N_\varepsilon(E) \rangle$ given in equation (9) and performing the integration (for $E \ll D/L^2_{x,y,z}$) will result in

$$\langle \delta^2 N_\varepsilon(E) \rangle_\varphi = \frac{1}{2\pi^2} \left[\ln\left(\frac{E^2}{\Gamma^2} + 1 \right) + \frac{L_x}{\frac{1}{2}\sin(\pi/8)} \left(\frac{E}{D} \right)^{1/2} \right]. \tag{14}$$

Thus, for energies below the Thouless energy the SNS junction phase-averaged variance will deviate from both the GUE and GOE statistics. For energies above the Thouless energy the order parameter phase has no influence on the variance and the functional form of the variance will correspond to equation (12).

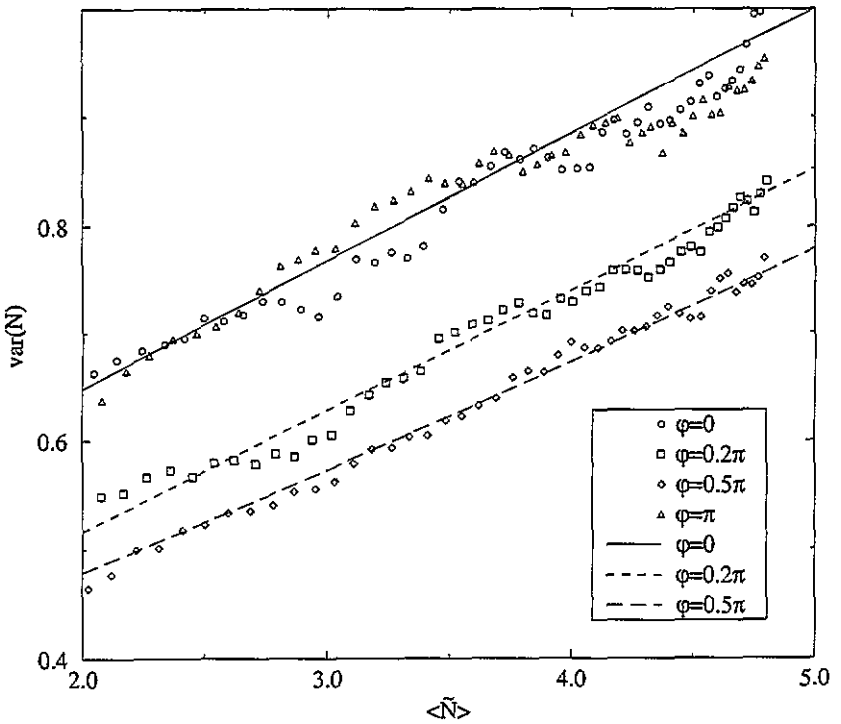

Figure 6. The variance of the number of levels for values of energy as a function of the normalized number of levels $\langle \tilde{N} \rangle$.

We have also performed exact diagonalization numerical studies of the SNS junction in order to verify the previous results. The BdG equation (equation (1)) can be written as a matrix in a tight-binding site representation [26] resulting in a matrix **M** of order $2n \times 2n$, where n is the number of sites in the SNS junction. $M_{i,j} = H_{i,j}$, where **H** is a $n \times n$ matrix corresponding to the tight-binding representation of the single-electron Hamiltonian (i.e., $H = \sum \epsilon_l \psi_l \psi_l^* + V \sum_{\{l,m\}} \psi_l \psi_m^*$, $\{l, m\}$ are nearest-neighbour sites) and $1 \leqslant i, j \leqslant n$. For sites belonging to the normal region the on-site energies assume values of $-W/2 < \epsilon_l < W/2$, while in the superconducting regions $\epsilon_l = 0$. In a similar fashion $M_{i+n,j+n} = -H_{i,j}$. The off-diagonal blocks of **M** are determined by the order parameter $M_{i+n,j} = \delta_{i,j} \Delta_i \exp(\mathrm{i}\chi_i)$, where $\delta_{i,j}$ is a Kronecker delta, $\Delta_i = \Delta_0$ and $\chi_i = \chi_{1(2)}$ for sites belonging to the left (right) superconducting region, while $\Delta_i = 0$ for sites in the normal region. Likewise $M_{i,j+n} = \delta_{i,j} \Delta_i \exp(-\mathrm{i}\chi_i)$.

The numerical calculations were performed for a two-dimensional SNS junction with 20×20 sites (i.e., matrices of size $2n = 800$). The first five columns of sites on each side of the junction are considered as the superconducting region, while the central region (of size 10×20) is normal. The disorder parameter was chosen as $W = 2V$ and the order parameter $\Delta_0 = V$. An order parameter $\Delta_0 = 4V$ was also investigated but no essential difference in the behaviour of the low-lying energy levels was observed. The averaging over different configurations of disorder was performed over 3000 samples.

First of all the dependence of the energy levels on the order parameter phases was investigated. As expected, the energy levels depend only on the phase difference φ and not on the individual values of χ_1 and χ_2. The energy levels of a single realization as well as the averaged energy levels of the SNS junction are periodic in φ with a period of 2π. In figure 3

the level spacing distribution $P(s)$ averaged over the 30 lowest electronic levels (levels 400–430) for different values of the order parameter phase differences ($\varphi = 0,\ 0.2\pi,\ 0.5\pi,\ \pi$) are presented. It can be seen that for $\varphi = 0, \pi$ the GOE level spacing distribution is followed while for $\varphi = 0.2\pi,\ 0.5\pi$ an almost perfect GUE distribution is obtained. Thus, for $P(s)$ the transition from GOE statistics to GUE statistics is almost completed already for $\varphi = 0.2\pi$.

The variance of the number of levels in a given energy region as a function of the averaged number of levels $\langle \delta^2 N_\varepsilon(E) \rangle$, centred around the 410th level, for values of energy smaller than the Thouless energy is shown in figure 4. From comparing the numerical results to the predictions of the RMT theory given in equation (11) it is clear that for the $\varphi = 0, \pi$ cases the GOE variance fits perfectly, while for the $\varphi = 0.5\pi$ case a reasonable fit to the GUE statistics is seen. For the $\varphi = 0.2\pi$ the behaviour is in between the two statistics. Thus, as was predicted in equation (9) the periodical dependence of the variance on φ has a period of π and exhibits a crossover from GOE to GUE statistics. It is interesting to note that the variance is more sensitive to the order parameter phase difference than the level spacing distribution.

As we have previously mentioned, in order to compare quantitatively the results of the diagrammatic perturbation calculation given in equation (9) to the numerical results one must be in the region of $N_\varepsilon(E) > 1$. This region is plotted in figure 5. The summation in equation (9), which describes the crossover between the GOE and GUE statistics as well as the non-universal behaviour due to the Thouless energy, has been performed numerically and corresponds to the curves in the figure. The value of the Thouless energy in that region is estimated from the fit to the $\varphi = 0.5\pi$ case as $D/L_x^2 = 6s$ and as previously discussed $\Gamma = s$. The constant λ_{GOE} is added for the $\varphi = 0, \pi$ cases while λ_{GUE} is added for $\varphi = 0.2\pi, 0.5\pi$. It can be seen that as long as $\langle N \rangle < 5$ there is a reasonable agreement between the numerical results and the theory, while for $\langle N \rangle > 5$ the numerical data is lower than theory. The reason for this discrepancy is the fact that we consider only a finite region of the low-lying electronic excitations (i.e., levels 400–420) in order not to venture into hole states from one side and not to reach into the high-lying excitations from the other side. As noted in [31], when one considers only a finite number of levels the variance as a function of $\langle N \rangle$ is lower than expected and one should renormalize the average number of levels in the following way: $\langle \tilde{N} \rangle = \langle N \rangle (1 - \langle N \rangle / N_0)$, where N_0 is the number of states considered. The Thouless energy should also be normalized in a similar fashion, $\tilde{D}/L_x^2 = D/L_x^2(1 - \langle N \rangle / N_0)$. The variance as a function of $\langle \tilde{N} \rangle$ is presented in figure 6. An excellent fit of the theory to the normalized level number is obtained.

In conclusion, the spectra statistical properties of an SNS junction were investigated analytically and numerically. The energy levels depend on the difference between the order parameter phases with a period of 2π. The statistical properties of the energy levels also depend on φ but with a periodicity of π. For $\varphi = 0$ the statistical properties follow the GOE statistics, while for $\varphi \neq 0$ they may be described as a combination between GOE and GUE statistics.

Acknowledgments

I would like to thank Y Avishai and B Ya Shapiro for useful discussions, the Alon Foundation, the internal grant program of the Bar-Ilan University and the US–Israel Binational Science Foundation for financial support.

References

[1] Gorkov L P and Eliashberg G M 1965 *Zh. Eksp. Teor. Fiz.* **48** 1407 (Engl. Transl. 1965 *Sov. Phys.–JETP* **21** 940)

[2] Altshuler B L and Shklovskii B I 1986 *Zh. Eksp. Teor. Fiz.* **91** 220 (Engl. Transl. 1986 *Sov. Phys. JETP* **64** 127)

[3] Sivan U and Imry Y 1987 *Phys. Rev.* B **35** 6074

[4] Evangelou S N and Economou E N 1992 *Phys. Rev. Lett.* **68** 361

[5] Izrailev F M 1990 *Phys. Rep.* **129** 299

[6] Shklovskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 *Phys. Rev.* B **47** 11 487

[7] Simmons B D and Altshuler B L 1993 *Phys. Rev.* B **48** 5422

[8] Wigner E P 1951 *Ann. Math.* **53** 36; 1955 *Ann. Math.* **62** 548; 1957 *Ann. Math.* **65** 203; 1958 *Ann. Math.* **67** 325

[9] Dyson F J 1962 *J. Math. Phys.* **3** 140, 1157, 166

[10] Metha M L 1991 *Random Matrices* (San-Diego, CA: Academic)

[11] Shklovskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 *Phys. Rev.* B. **47** 11 487

[12] Harris R and Yan Z 1993 *J. Phys.: Condens. Matter* **5** L493

[13] Hofstetter E and Schreiber M 1993 *Phys. Rev.* B **48** 16 979

[14] Kravtsov V E, Lerner I V, Altshuler B L and Aronov A G 1994 *Phys. Rev. Lett.* **72** 888

[15] Evangelou S N 1994 *Phys. Rev.* B **49** 16 805

[16] Dupuis N and Montambaux G 1991 *Phys. Rev.* B **43** 14 390

[17] Huckestein B and Schewitzer L 1992 *High Magnetic Fields in Semiconductor Physics III* ed G Landwehr (Berlin: Springer)

[18] Ono Y, Kuwano H, Slevin K, Ohtsuki T and Kramer B 1993 *J. Phys. Soc. Japan* **62** 2762

[19] Ono Y and Ohtsuki T 1993 *J. Phys. Soc. Japan* **62** 3813

[20] Eliyahu D and Berkovits R 1994 *J. Phys.: Condens. Matter* **6** 4737

[21] Spivak B Z and Khmelnitskii D E 1982 *JETP Lett.* **35** 413

[22] Altshuler B L and Spivak B Z 1987 *Sov. Phys.–JETP* **65** 343

[23] Beenakker C W J and van Houten H 1991 *Phys. Rev. Lett.* **66** 3056

[24] Glazman L I and Matveev K A 1989 *JETP Lett.* **49** 659

[25] Beenakker C W J 1991 *Phys. Rev. Lett.* **67** 3863

[26] Hui V C and Lambert C J 1993 *Europhys. Lett.* **23** 203

[27] Lambert C J 1993 *J. Phys.: Condens. Matter* **5** 707; 1991 *J. Phys. Condens. matter* **3** 6759

[28] Andreev A F 1964 *Sov. Phys.–JETP* **19** 1228

[29] de Gennes P G 1966 *Superconductivity of Metals and Alloys* (New York: Benjamin)

[30] Berry M V and Robnick M 1986 *J. Phys. A: Math. Gen.* **19** 649

[31] Altshuler B L, Zharekeshev I Kh, Kotochigova S A and Shklovskii B I 1988 *Zh. Eksp. Teor. Fiz.* **94** 343 (Engl. Transl. 1988 *Sov. Phys.–JETP* **67** 625)